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Critical Behavior of Sandpile Vortices under Variable Charge

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In the following, we consider a sandpile cellular automaton model (height version), which also takes account of lattice cyclicity for variable charge of grains projected upon the board. The size of the "charge" or additional grains of sand used to upset the equilibrium and to induce an avalanche is found to affect the distributional forms and in particular the convergence of the estimate for the dynamic exponent. Exponent estimates show slow variation with lattice dimension L and some evidence of evolution with charge. The scaling region is limited in all cases with noisy decay at all levels of charge. Results presented extend the work of Duarte and Goncalves (1990) on the triangular lattice.

KEY WORDS: Cellular Automata; sandpiles; height model; vortices; lattice cyclicity.

1. INTRODUCTION

Although a precise definition of self-organized criticality (SOC) still appears to be lacking, the principal characteristic appears to be the existence of scale-independent fluctuations without the fine-tuning.[21,3] The cellular automaton models of Bak *et al.*[1] and Wiesenfeld *et al.*[23] provide a means of demonstrating SOC through a transport system consisting of a pile of sand with the flow of grains representing the order parameter. The authors argued that such dynamical systems show natural stochastic time evolution (t large) to a unique critical state, achieved without varying the system control parameters and independent of the initial conditions. It was expected that this "self-organization" process would be characterized by power-law correlations in space and time. Despite the relative simplicity

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and wide applicability of these models, there is some evidence that power laws are not wholly adequate to describe the observed behavior.[11,14]

Three principal models of sandpiles currently form the basis for discussion in the literature, namely the critical height, critical slope, and Laplacian models.[14] Of these the first has attracted most attention and in the general height model, the SOC state has been shown to be unaffected by changes in the toppling rules from site to site and in any dimension, although some stable configurations are forbidden.[16,2] Despite the fact that the "slope" model does not in fact describe the slope as such but rather local height differences (e.g., ref. 14), it provides a more natural representation of the dynamics of toppling than absolute heights. The attraction of the height model is its simple mathematical structure[2] and it has proved useful as a "toy model" for complex systems, despite being criticized as unrealistic in terms of true sandpile behavior.[15] Recent important work on the molecular dynamics simulation of granular flow (e.g., refs. 8, 19, 10, and 17 and references therein) has also failed to show a clear connection between these more realistic models and SOC. Nevertheless, a number of both isotropic and anisotropic variants of the height model have been proposed (see, e.g., refs. 18 and 20) and various applications of current importance have been studied, notably turbulence as a model for hydrodynamic instability (ref. 7 and references therein) and forest fires.[6] Recently, efforts have also been made to model natural sandpile behavior more closely and Ding *et al.*[5] have considered a stochastic slide restricted to a part of the sample with sand lost through the opposite side.

In their work on turbulence, Duarte and Goncalves[7] describe the propagation of avalanches on a class of "cyclic" lattices which provide eddies of all sizes up to the limit of the linear dimension. Of interest also in terms of the formation of these eddies or vortices is the effect on the avalanche and cycle distributions of varying the amount of "charge" or quantity of sand projected onto the board. In what follows, we therefore describe an extension of the work for the triangular lattice over a range of lattice dimension L for the case of variable charge.

2. TURBULENCE AND THE HEIGHT MODEL

The height model on a regular lattice, as is now well known, describes the occupation of lattice sites by columns of sand grains which remain stable until a critical value h_c is reached. If h_c is exceeded, the excess grains topple and are shared equally among neighboring sites, i.e.,

$$h(i, j) \rightarrow h(i, j) - h_c \tag{1}$$

where $h(i, j)$ is the site occupation value. If the toppling and sharing process induces criticality in neighboring sites, then they, too, will topple and an avalanche is generated with average size and duration independent of the initial conditions. The distributions for average size s and lifetime t, weighted by the average response, are expected to obey

$$D(s) \sim s^{1-\tau} \tag{2}$$

$$D(t) \sim t^{-b'} \tag{3}$$

with exponents $1-\tau$ and b', respectively.[13,7] Majumdar and Dhar[12] have recently given revised estimates for these exponents which are slightly lower than those previously obtained, although still in reasonable agreement with earlier work. These authors quote $\tau = 8/7$ and the simple decay exponent $b = 19/15$, respectively. The first moment exponent b' remains controversial, with two very different values quoted by Bak *et al.*[1] and Manna.[13] We shall be concerned with the estimation of these exponents and the form of the observed distributions as the charge is varied.

In standard SOC models, backfiring is always possible, since any site which is a neighbor to one which has previously toppled may still redistribute excess sand to all neighbors, so that more than one avalanche per site can occur. The absence of this phenomenon leads to some simplification for anisotropic models, where some exact results are known,[4] but the corresponding extension to the isotropic case is still lacking. Backfiring is also prohibited on cyclic lattices, but, unlike the fully-directed case, feedback information which is important in turbulent behavior is retained.

3. RESULTS

The geometrical configuration and conditions which we adopt are those described by Duarte and Goncalves,[7] but multispin coding has not been used. The indications are that the technique is not particularly efficient given the concentrated area of the avalanches and the small number at the transition. Consequently, the lookup table is subject to permanent updates, but saves on the long lattice update (e.g., ref. 9) and provides a real reduction in time. Triangular lattices of size 20–200 have been considered with charges varying between 1 and 20 for number of samples ranging from 10^5 to 10^6. In general, we found distributions to be "noisy," with a comparatively short useful range where scaling behavior as defined in (2) and (3) above could be clearly demonstrated. Consequently, log–log fits refer to the truncated range in all cases, although individual small-scale fluctuations may be included. We found that a distribution of

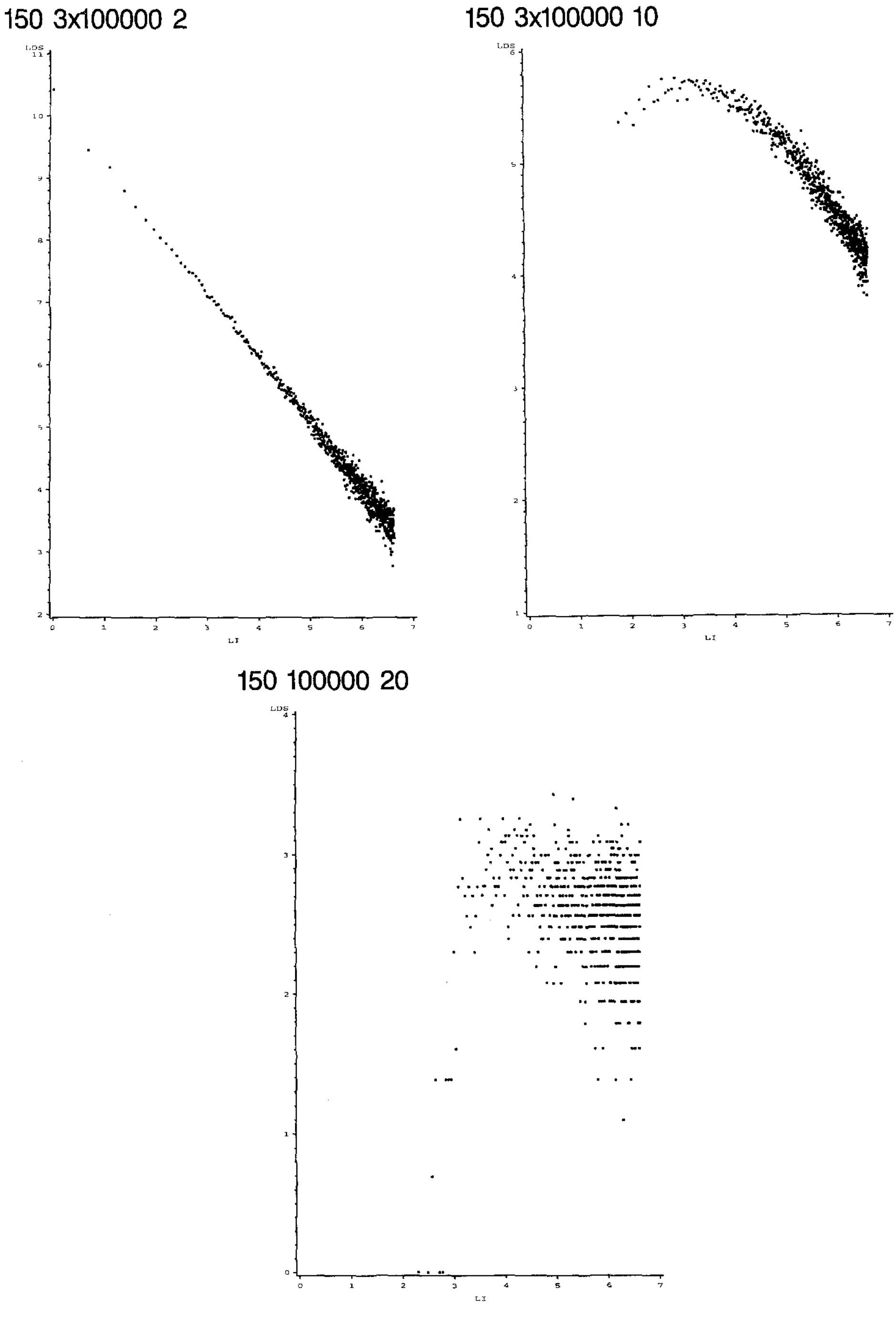


Fig. 1. Avalanches for variable charge: Numbers heading individual graphs indicate lattice dimension, number of trials, and size of charge, respectively. Key: LDS $= \ln(D(s))$; LI $= \ln(s)$.

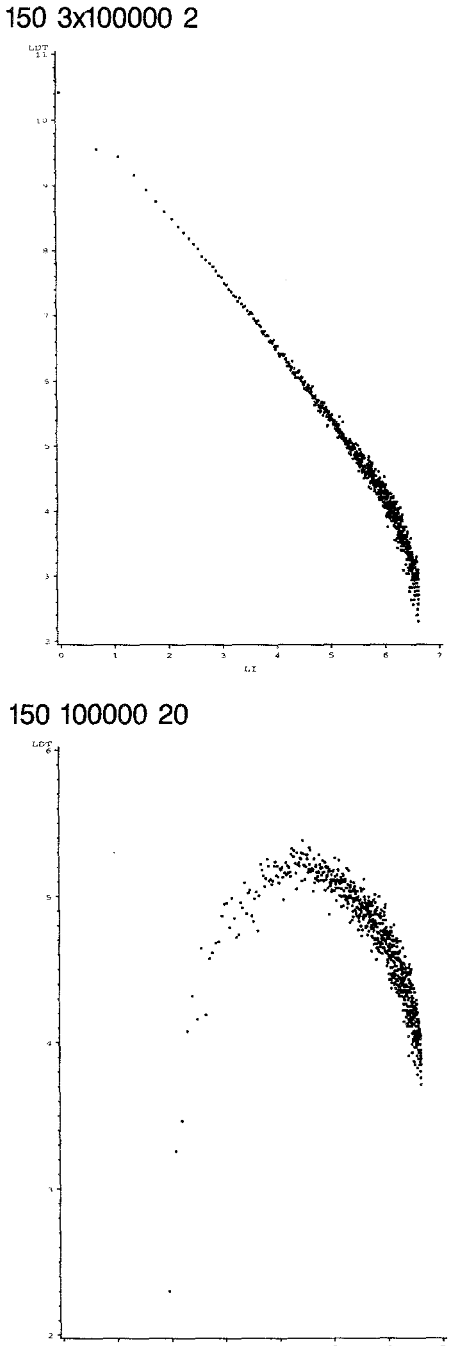
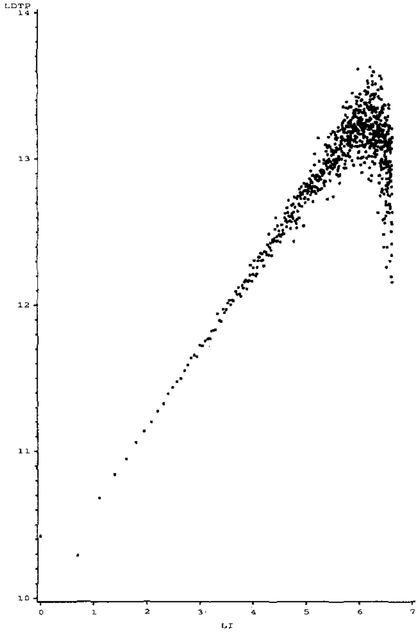


Fig. 2. Average lifetime distribution for variable charge: Numbers heading individual graphs are as for Fig. 1. Key: LDT = log of unweighted lifetime distribution; LI = ln(t).

150 3x100000 2



150 100000 20

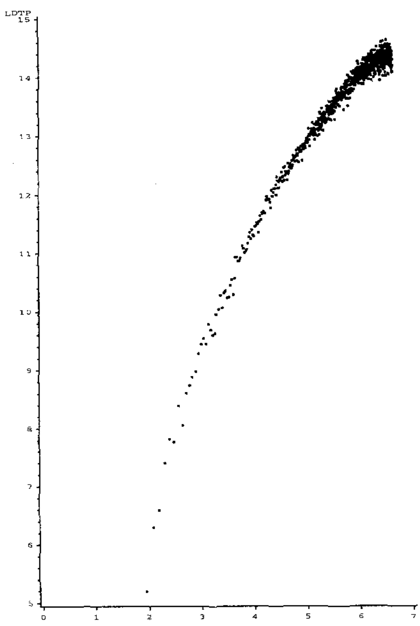


Fig. 3. Weighted lifetime distributions for variable charge: Numbers heading individual graphs are as for Fig. 1. Key: LDTP $= \ln(D(t))$; LI $= \ln(t)$.

cycle or eddy sizes was only achieved by introducing fairly large charges of grains to the board (in excess of 5 for most lattice sizes considered). Introducing a larger charge, however, was accompanied by a shift in the modal peak of the average lifetime distribution corresponding to that observed in widely used lifetime distributions such as the Weibull[22] or its logarithmic counterpart, the extreme value distribution. This was particularly marked for larger lattice sizes, where effects may be assumed to give

Table I. Height Model with Cyclicity: Exponent Ranges under Variable Charge[a]

Lattice dimension L	Charge size	Number of trials	Range of b	Range of τ	Range of b'
20	1	10^6	−1.105 to −1.319	−1.016 to −1.026	+0.484 to +0.501
20	10	10^6	−0.876 to −1.433	−0.401 to −0.421	+1.719 to +2.035
20	20	10^6	(−5.204 to −8.546)	−0.170 to −0.194	Not estimable
30	1	10^6	−1.320 to −1.380	−1.018 to −1.035	+0.423 to +0.503
30	3	10^6	−1.106 to −1.205	−0.894 to −0.947	+0.582 to +0.669
30	4	10^6	−0.719 to −1.163	−0.764 to −0.860	+0.792 to +0.848
30	10	10^6	−0.454 to −1.309	−0.323 to −0.528	+1.589 to +1.704
50	1	10^6	−1.012 to −1.377	−1.025 to −1.067	+0.497 to +0.507
50	2	10^6	−0.953 to −1.006	−0.994 to −1.021	+0.545 to +0.560
50	3	10^6	−0.864 to −0.916	−0.909 to −0.965	+0.626 to +0.648
50	5	10^6	−0.768 to −1.117	−0.528 to −0.841	+0.762 to +0.815
50	10	10^6	−0.673 to −0.888	−0.449 to −0.542	+1.323 to +1.851
50	20	3×10^5	(−1.768 to −4.148)	~(−0.2 to −0.6)*	+2.063 to +2.289
80	10	10^6	−0.454 to −0.596	−0.574 to −0.508	+1.269 to +1.723
80	20	10^6	(−3.630 to −5.509)	~(−0.2 to −0.5)*	+1.738 to +2.114
150	1	6×10^5	−1.038 to −1.157	−1.058 to −1.086	+0.454 to +0.470
150	2	3×10^5	−0.964 to −1.075	−1.027 to −1.045	+0.525 to +0.529
150	4	6×10^5	−0.739 to −0.944	−0.771 to −0.899	+0.699 to +0.792
150	5	10^6	−0.715 to −0.885	−0.550 to −0.860	+0.725 to +0.970
150	10	3×10^5	−0.392 to −0.716	−0.498 to −0.606	+0.884 to +1.736
150	20	3×10^5	−0.477 to −0.690	−0.139 to −0.207*	+1.340 to +2.502
200	1	10^5	−1.059 to −1.119	−1.063 to −1.093	+0.433 to +0.457
200	5	10^5	−0.657 to −0.808	−0.624 to −0.780	+0.784 to +1.038

[a] Estimation range is relatively unstable since, as the number of points included increases, so does the slope in general. In part this is due to the fluctuations noted, since these would normally be included until oscillations become persistent in order to achieve a reasonable length sequence. The value of the correlation coefficient measure is therefore limited, partly because of the skewness of the underlying distributions and partly due to the inherent autocorrelation. Asterisks indicate poor fits, noted in particular for large charge and "smeared" distributions (see text), and refer to especially poor correlation measures.

a truer representation of asymptotic behavior. Such an analogy suggests that modifications of the simple power-law behavior to include a shape parameter may well be appropriate. For very large charge, the modal peak is effectively flattened or smeared over a considerable range of $\ln(\langle s \rangle)$ or $\ln(\langle t \rangle)$, respectively, and typical avalanche behavior is both shifted and reduced. We note again that distributions are noisy with fluctuations appearing at relatively low values of $\langle s \rangle$ and $\langle t \rangle$. We illustrate for lattice size 150 and various levels of charge in Figs. 1–3.

In Table I, we summarize results obtained on log–log least squares fits over the scaling region for various lattice sizes and levels of charge. Our results for the dynamic exponent b show slow variation with L and some evidence of evolution with charge. We note similar behavior with respect to the static exponent τ, so that as charge increases, both exponents decrease, although not smoothly and with large uncertainties. The dynamic exponent here is the decay exponent and for charge 1 in particular seems to be in very reasonable agreement with the value obtained by Duarte and Goncalves[7] and Majumdar and Dhar.[12] Our τ value for charge 1 would appear to be slightly lower than that quoted by these authors for smaller lattice dimensions, but taking all L into account, a value of $\tau = 8/7$ seems quite plausible. For the other dynamic exponent, as defined in Eq. (2), our results for charge 1 appear to be much closer to the value quoted by Bak

Table II. Cycle Distributions: Average Eddy Size under Variable Charge[a]

Lattice Dimension L	Charge	Average eddy size
20	1, 2, 4	0
	10	0.499
	20	3.160
30	1, 3, 4	0
	10	0.548
50	1, 2, 3, 5	0
	10	0.584
	20	4.039
80	10	0.604
	20	4.261
150	1, 2, 4, 5	0
	10	0.619
	20	4.451

[a] Results from fitting a form $A + Bc^x$, with c = charge, indicate $x \sim -1.0$.

et al.[1] than to that of Manna,[13] but are again dependent on charge and as this increases, the exponent attains values not incompatible with that quoted by Manna and even higher. These values should be interpreted cautiously, however, since estimate ranges are unstable. It is noticeable once again that variation with L appears to be small.

Average eddy or cycle size is examined in Table II and shows slow evolution with L, but a large push appears to be necessary before any spread of eddy sizes is achieved. It seems clearly indicated that this average varies also with level of charge as expected, with preliminary fits indicating an exponent $x \sim -1.0$.

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REFERENCES

1. P. Bak, C. Tang, and K. Weisenfeld, *Phys. Rev. A* **38**:364 (1988).
2. D. Dhar, *Phys. Rev. Lett.* **64**:1613 (1990).
3. D. Dhar and S. N. Majumdar, *J. Phys. A* **23**:4333 (1990).
4. D. Dhar and R. Ramaswamy, *Phys. Rev. Lett.* **63**:1659 (1989).
5. E. J. Ding, Y. N. Lu, and H. F. Ouyang, *Phys. Rev. A* **46**:R6136 (1992).
6. B. Drossel and F. Schwabl, *Phys. Rev. Lett.* **23**:1629 (1992).
7. J. A. M. S. Duarte and N. J. A. P. Goncalves, *Physica A* **168**:901 (1990).
8. J. A. C. Gallas, H. J. Herrmann, and S. Sokolowski, *Phys. Rev. Lett.* **69**:1371 (1992).
9. P. Grassberger and S. S. Manna, *J. Phys. (Paris)* **51**:1077 (1990).
10. D. C. Hong and J. A. McLennan, *Physica A* **187**:159 (1992).
11. J. H. Jensen, K. Christensen, and H. C. Fogedby, *Phys. Rev. B* **40**:7425 (1989).
12. S. N. Majumdar and D. Dhar, *Physica A* **185**:129 (1991).
13. S. S. Manna, *J. Stat. Phys.* **59**:509 (1990).
14. S. S. Manna, *Physica A* **179**:249 (1991).
15. A. Mehta, *Physica A* **186**:121 (1992).
16. S. P. Obukhov, in *Random Fluctuations and Pattern Growth*, H. E. Stanley and N. Ostrowsky, eds. (Dordrecht, Kluwer, 1988).
17. T. Poschel, *J. Phys. II (Paris)* **3**:27 (1993).
18. H. Puhl, *Physica A* **182**:295 (1991).
19. G. H. Ristow, *J. Phys. I (Paris)* **2**:649 (1992).
20. B. Tadic, U. Nowak, K. D. Usadel, R. Ramaswamy, and S. Padlewski, *Phys. Rev. A* **45**:8536 (1992).
21. H. Takasayu, I. Nishikawa, and H. Tasaki, *Phys. Rev. A* **37**:3110 (1988).
22. W. Weibull, *J. Appl. Mech.* **18**:295 (1951).
23. K. Wiesenfeld, C. Tang, and P. Bak, *J. Stat. Phys.* **54**:1441 (1989).

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